DM 'Partiel': Stone–Čech compactification

Let X be a *Tychonoff space,* this means it satisfies the separation axioms (T_1) and

$$(T_{3\frac{1}{2}}): \quad \begin{array}{l} \text{for each } x \in X \text{ and a closed set } F \subset X \text{ s.t. } x \notin F \\ \text{there exists a continuous function } f : X \to [0,1] \\ \text{such that } x \in f^{-1}(\{0\}) \text{ and } F \subset f^{-1}(\{1\}). \end{array}$$

0. Recaps, basics.

(i) Quote (T_1), (T_2), (T_4), and Urysohn's lemma; argue that *normal Hausdorff spaces* (i.e., topological spaces satisfying (T_1) and (T_4)) are Tychonoff spaces.

(ii) Prove that *compact Hausdorff spaces* (='compacts en français' = compact topological spaces satisfying (T_2)) are normal Hausdorff (i.e., satisfy (T_4)).

Definition. A *Stone–Čech compactification* of a topological space X is a compact Hausdorff space βX and a continous mapping $\iota = \iota_X : X \to \beta X$ such that the following universal property holds:

> for each continuous mapping $f : X \to K$ of the space X to a compact Hausdorff space K there exists a unique continuous mapping $\beta f : \beta X \to K$ such that $\beta f \circ \iota_X = f$.

(iii) Prove that the space βX, if exists, is unique up to a homeomorphism.

(iv) Provided X is a Tychonoff space, prove that $\iota : X \to \iota(X)$ has to be a bijection and, moreover, a *homeomorphism* (the topology on $\iota(X)$ is induced from βX). [*Hint:* For the latter, prove that $\iota(F)$ is closed in $\iota(X)$ if F is closed in X.]

Informally speaking, we aim to homeomorphically embed a Tychonoff space X into a (huge) compact Hausdorff space such that all continuous functions $f : X \to K$ admit(!) a unique(!) continuation onto this bigger space.

Our first goal is to prove that, for Tychonoff spaces X, the Stone–Çech continuation βX *exists* by giving an explicit construction based upon Tychonoff's theorem.

1. Construction. Denote $C_X := C(X; [0,1])$, the space of all continuous functions from X to $[0,1]$, and consider a *Tychonoff cube*

$$K(C_X) := [0,1]^{C_X} = \prod_{x \in X} [0,1] = \{\, \Phi : C_X \ni g \mapsto \Phi(g) \in [0,1] \,\}$$

(where one does *not* assume any property of Φ), equipped with the usual product topology. Recall that $K(C_X)$ is a compact space due to the Tychonoff theorem.

Now consider the mapping $I : X \to K(C_X)$, $x \mapsto I(x)$, where the *evaluation functional* $I(x) \in K(C_X)$ is defined as follows: $[I(x)](g) := g(x)$.

(i) Prove that $I : X \to I(X)$ is a bijection. Recall the definition of the topology on $I(X) \subset K(C_X)$ and use it to define a 'new' topology on $X = I^{-1}(I(X))$.

(ii) Using the fact that X is a Tychonoff space, prove that this 'new' topology is the same as the original one and thus $I : X \to I(X) \subset K(C_X)$ is a homeomorphism.

Let us *define* $\beta X := \overline{I(X)}^{K(C_X)}$, the closure of $I(X)$ in the topology of $K(C_X)$.

(iii) Argue that thus defined βX is a compact Hausdorff space.

2. Universal property. Assume now that $f : X \to K$ is a continuous function, where K is a compact Hausdorff space. We aim to prove that there exists a unique continuous function $\beta f : \beta X \to K$ s.t. $f = \beta f \circ I$, where βX is constructed above.

(i) Argue that such a function $\beta f : \beta X \to K$, if exists, is unique.

(ii) Consider first the case $K = [0,1]$. Given a continuous function $f : X \to [0,1]$, prove that the function $\beta f : K(C_X) \to [0,1]$, $\Phi \mapsto \Phi(f)$, is continuous and that its restriction onto $\beta X \subset K(C_X)$ satisfies the required property $f = \beta f \circ I$.

(iii) Now consider the case when $K = K(A) = [0,1]^A$ is also a Tychonoff cube (we do not make any assumption on A here). Again, prove that each continuous function $f : X \to K(A)$, $x \mapsto f(x) : A \to [0,1]$, admits a *continuous(!)* extension $\beta f : K(C_X) \to K(A)$ defined as follows: $[(\beta f)(\Phi)](\alpha) = \Phi(f(\cdot)(\alpha))$.

(iv) Finally, argue that each compact Hausdorff space K can be homeomorphically embedded into a Tychonoff cube $K(A)$ with $A = C_K$ and conclude the proof.

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3 Bonus: ultrafilters on $\mathbb{N}$.** It is easy to see that $\beta X \cong X$ if X is a finite set. Let us consider the simplest nontrivial example: $X = \mathbb{N}$ (equipped with the discrete topology). Recall that $\mathcal{U} \subset 2^{\mathbb{N}} \smallsetminus \{\emptyset\}$ is called a (proper) *ultra-filter* if

- for each $Y \subset \mathbb{N}$ either $Y \in \mathcal{U}$ or $\mathbb{N} \smallsetminus Y \in \mathcal{U}$ (but not both);
- if $Y \in \mathcal{U}$, then $Y' \in \mathcal{U}$ for all larger sets $Y \subset Y' \subset \mathbb{N}$;
- if $Y_1, Y_2 \in \mathcal{U}$, then $Y_1 \cap Y_2 \in \mathcal{U}$.

Prove that the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ is homeomorphic to the set of all proper ultra-filters on $\mathbb{N}$ equipped with the *Stone topology*, i.e., the topology generated by the base sets $\mathcal{W}_Y := \{\mathcal{U} : Y \in \mathcal{U}\}$, where Y runs over all subsets of $\mathbb{N}$. The inclusion $\iota : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ is defined as $n \mapsto \mathcal{U}_n := \{Y \subset \mathbb{N} : n \in Y\}$.

(i) Check that the Stone topology is correctly defined (i.e., that one can use the family $\{\mathcal{W}_Y\}_{Y\subset\mathbb{N}}$ as a base set to define a topology) and that it is Hausdorff.

(ii) Prove that thus defined topological space is compact. [*Hint:* This is equivalent to the following statement: given a family of sets Y_α such that no finite sub-family covers $\mathbb{N}$, one can define an ultrafilter on $\mathbb{N}$ that does not contain any of Y_α.]

(iii) Study the notion of *ultra-limits* and prove that, given an arbitrary function $f : \mathbb{N} \to K$, one can set $(\beta f)(\mathcal{U}) := \mathcal{U}\text{-}\lim f(n)$ and that $\beta f : \beta\mathbb{N} \to K$ is continuous (where $\beta\mathbb{N}$ is now defined as the topological space of all proper ultra-filters on $\mathbb{N}$).
